New hyperfinite subfactors with infinite depth

Julio E. Cáceres Gonzales
(joint with Dietmar Bisch)

Vanderbilt University

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Outline

1. Preliminaries
   - Subfactors
   - Motivation
   - Commuting squares
   - Planar algebras
2. New A_∞-subfactors
   - Main idea
   - New commuting squares
   - Intermediate commuting squares

Subfactors

II_1 factor: an infinite dimensional von Neumann algebra with a unique normal tracial state tr and trivial center. It is *hyperfinite* if it can be approximated by finite dimensional C*-algebras.

Subfactor $N \subset M$: a unital inclusion of II_1 factors. It is *irreducible* if $\dim N' \cap M = 1$.

Standard representation $L^2(M, \mathrm{tr})$: the GNS representation of M with respect to the trace tr. The orthogonal projection $e_N : L^2(M) \to L^2(N)$ is called the *Jones projection*.

Jones Index $[M : N]$: measures how "big" M is relative to N, can be shown it is equal to $\mathrm{tr}(e_N)^{-1}$.

Basic construction: $N \subset M \subset \langle M, e_N \rangle \subset B(L^2(M))$. $M_1 := \langle M, e_N \rangle$ will be a II_1 factor precisely when $[M : N] < \infty$.

Subfactor invariants

Whenever $[M : N] < \infty$ we can iterate the basic construction and get the Jones tower $N \subset M \subset M_1 \subset M_2 \subset \cdots \subset \overline{\cup_{n\geq 1} M_n}^{w} = M_\infty$

Standard invariant: is the lattice of inclusions

$$\begin{array}{ccccccccccc} N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & \cdots & \subset & N' \cap M_n & \subset & \cdots \\ & & \cup & & \cup & & & & \cup & & \\ & & M' \cap M & \subset & M' \cap M_1 & \subset & \cdots & \subset & M' \cap M_n & \subset & \cdots \end{array}$$

Principal and dual principal graphs: they are two graphs (Γ, Γ') that describe the top and bottom tower of inclusions in the standard invariant. We say $N \subset M$ has finite depth if Γ is finite.

When $\Gamma = \Gamma' = A_\infty = \circ\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!- \cdots$ we say $N \subset M$ has *trivial standard invariant* (also referred as A_∞-subfactor).

Open questions

1. What are all possible standard invariants for hyperfinite irreducible subfactors?
2. What are all hyperfinite subfactors $N \subset M$ with small index?
3. What is $\{[M : N], N \subset M$ hyperfinite, $N' \cap M = \mathbb{C}\}$? (Jones'83)

Open questions

1. What are all possible standard invariants for hyperfinite irreducible subfactors?
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Theorem (Jones rigidity [Jon83])

Consider a unital inclusion $N \subset M$ of II_1-factors, then

$$[M : N] \in \left\{4\cos^2\left(\frac{\pi}{n}\right),\ n \geq 3\right\} \cup [4, \infty].$$

Moreover, every index is attained by some hyperfinite subfactor.

Landscape (finite depth)

- $[M : N] \leq 4$: ADE classification
- $4 < [M : N] < 5.25$: classification of small index subfactors (work of many people)

Index	# of subfactors	Name
$\frac{1}{2}(5+\sqrt{13})$	2	Haagerup
≈ 4.37720	2	Extended Haagerup
$\frac{1}{2}(5+\sqrt{17})$	2	Asaeda-Haagerup
$3+\sqrt{3}$	2	3311
$\frac{1}{2}(5+\sqrt{21})$	2	2221
5	7	-
≈ 5.04892	2	$\mathfrak{su}(2)_5$ and $\mathfrak{su}(3)_4$
$3+\sqrt{5}$	11	-

What about infinite depth?

- Irreducible hyperfinite subfactor at index ≈ 4.026418.
- If $4 < [M : N] < 3 + \sqrt{5}$ then it has trivial standard invariant, i.e. it is an A_∞-subfactor.
- Schou constructed many other irreducible hyperfinite subfactors coming from **commuting squares** with index different from previous ones.

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Conjecture (Bisch)

Every index of a hyperfinite finite depth irreducible subfactor is the index of a hyperfinite A_∞ subfactor.

Commuting squares

$$\begin{matrix} A_{1,0} & \overset{K}{\subset} & A_{1,1} \\ \cup_G & & \cup_L \\ A_{0,0} & \overset{H}{\subset} & A_{0,1} \end{matrix}$$

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Commuting squares

$$
\left.\begin{array}{ccccccccc}
A_{1,0} & \overset{K}{\subset} & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset & A_{1,\infty} \\
\cup_G & & \cup_L & & \cup & & & & \cup \\
A_{0,0} & \overset{H}{\subset} & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset & A_{0,\infty}
\end{array}\right\} \text{subfactor}
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They satisfy:

- $[A_{1,\infty} : A_{0,\infty}] = \|G\|^2 = \|L\|^2$
- Always hyperfinite
- Irreducible if G (or L) satisfy Wenzl's criterion

Constructing commuting squares

Theorem (Ocneanu's bi-unitary condition [Sch90])

The following are equivalent:

- *There exists a commuting square* $\begin{array}{ccc} A_{1,0} & \overset{K}{\subset} & A_{1,1} \\ \cup_G & & \cup_L \\ A_{0,0} & \overset{H}{\subset} & A_{0,1} \end{array}$.
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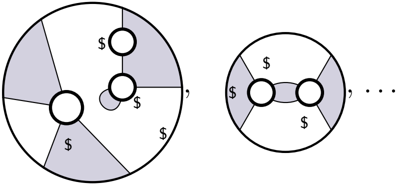
- *There exists a bi-unitary connection on the square of inclusions.*

This reduces the problem of constructing a commuting square to solving a large system of non-linear equations.

Problem: These equations vary wildly whenever you change G, H, K, L.

Planar algebras

A *planar algebra* is a collection of vector spaces $\{P_{n,\pm}\}$ which admit actions by diagrams like this:



Two important examples by V. Jones:

- Given $N \subset M$ an irreducible finite index subfactor, the standard invariant has a planar algebra structure. We denote this planar algebra by $P_\bullet^{N\subset M}$.
- Given G a bipartite graph, the loop spaces on G have a planar algebra structure. We denote this planar algebra by $\mathrm{GPA}(G)_\bullet$.

Module graphs and embeddings

Theorem (Grossman, Morrison, Penneys, Peters, Snyder '23)

Suppose $P_\bullet$ is a finite depth subfactor planar algebra. Let $\mathcal{C}$ denote the unitary multifusion category of projections in $P_\bullet$, with distinguished object $X = \mathrm{id}_{1,+} \in P_{1,+}$, and the standard unitary pivotal structure with respect to X. There is an equivalence between:

1. ***Planar algebra embeddings*** $P_\bullet \hookrightarrow \mathrm{GPA}(G)_\bullet$*, and*
2. *indecomposable finitely semisimple pivotal left $\mathcal{C}$-module C^* categories $\mathcal{M}$ whose fusion graph with respect to X is G (we call such G a* ***module graph****).*

Embedding theorem

Using Ocneanu's compactness, some facts about Pimsner-Popa basis and loop algebra formulas we prove the following:

Theorem (Bisch, C'22)

Let $P_\bullet$ be the subfactor planar algebra associated to $A_{0,\infty} \subset A_{1,\infty}$ and $\mathrm{GPA}(G)_\bullet$ the graph planar algebra associated to the inclusion graph of $A_{0,0} \subset A_{1,0}$. Then $P_\bullet$ embeds into $\mathrm{GPA}(G)_\bullet$.

$$\begin{array}{ccccccccc} A_{1,0} & \overset{K}{\subset} & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset & A_{1,\infty} \\ \cup_G & & \cup_L & & \cup & & & & \cup \\ A_{0,0} & \overset{H}{\subset} & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset & A_{0,\infty} \end{array}$$

Main idea

Let $N \subset M$ be a finite depth hyperfinite subfactor and G a graph that is not a module graph for $N \subset M$, i.e. $P_\bullet^{N \subset M} \not\hookrightarrow GPA(G)_\bullet$.

If $A_{0,\infty} \subset A_{1,\infty}$ comes from a commuting square $\begin{array}{ccc} A_{1,0} & \overset{K}{\subset} & A_{1,1} \\ \cup_G & & \cup_L \\ A_{0,0} & \overset{H}{\subset} & A_{0,1} \end{array}$ we get $P_\bullet^{A_{0,\infty} \subset A_{1,\infty}} \hookrightarrow GPA(G)_\bullet$. Hence, $A_{0,\infty} \subset A_{1,\infty}$ is not isomorphic to $N \subset M$.

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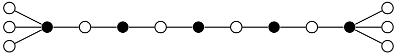
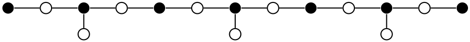
We know a lot about the module graphs for:

- Peters: Haagerup subfactor (3 graphs)
- GMPPS: Extended Haagerup subfactor (4 graphs)
- We computed 14 *potential* module graphs for the Asaeda-Haagerup subfactor using combinatorial data obtained by Grossman, Izumi and Snyder.

Non-module graphs

Large double broom $\|\cdot\|^2 = \frac{5+\sqrt{17}}{2}$	
Quipu $\|\cdot\|^2 \approx 4.37720$	

New A_∞-subfactors

Theorem (Bisch, C'23)

If G is one of the previous graphs, there exist H, K and L for which we can construct a bi-unitary connection.

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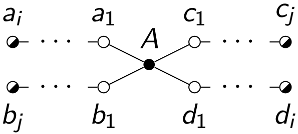
If G is one of the previous graphs, there exist H, K and L for which we can construct a bi-unitary connection.

We have constructed irreducible hyperfinite subfactors with indices $\frac{5+\sqrt{17}}{2}$ (the same as Asaeda-Haagerup) and ≈ 4.37720 (the same as Extended-Haagerup), by our graph planar algebra embedding and classification they must be A_∞-subfactors.

Corollary (Bisch, C'23)

There are hyperfinite A_∞-subfactors with indices $\frac{5+\sqrt{17}}{2}$ and ≈ 4.37720.

More connections

$S(i,i,j,j) =$ 

Let $G = S(i,i,j,j)$, the 4-star with two pairs of legs of equal length.

Theorem (Bisch, C'23)

There exists a 1-parameter family of non-equivalent bi-unitary connections

for all i,j for inclusions of the form
$$\begin{array}{ccc} A_{1,0} & \overset{G^t}{\subset} & A_{1,1} \\ \cup_G & & \cup_{G^t} \\ A_{0,0} & \overset{G}{\subset} & A_{0,1} \end{array}.$$

Another approach to show infinite depth

- Kawahigashi proved that given a finite depth subfactor $N \subset M$, there are only countably many non-equivalent commuting squares from which the subfactor can be constructed.
- By classification of small index subfactors we have finitely many finite depth subfactors at the indices $\frac{5+\sqrt{17}}{2}$, $3+\sqrt{3}$, $\frac{5+\sqrt{21}}{2}$, 5 and $3+\sqrt{5}$.
- Our 1-parameter families of non-equivalent bi-unitary connections must produce at least one infinite depth subfactor.

Indices of $S(i,i,j,j)$

$j \backslash i$	1	2	3	4	$\cdots$	∞
1	4					
2	$\frac{5+\sqrt{17}}{2}$	5				
3	$3+\sqrt{3}$	5.1249	$3+\sqrt{5}$			
4	$\frac{5+\sqrt{21}}{2}$	5.1642	5.2703	$\frac{7+\sqrt{13}}{2}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
∞	$2+2\sqrt{2}$	5.1844	5.2870	5.3184		$\frac{16}{3}$

Hence we have hyperfinite infinite depth subfactors at $\frac{5+\sqrt{17}}{2}$, $3+\sqrt{3}$, $\frac{5+\sqrt{21}}{2}$, 5 and $3+\sqrt{5}$. By classification, all but the last one must be A_∞-subfactors.

Intermediate commuting squares

$$\begin{array}{ccccccccc} A_{1,0} & \overset{K}{\subset} & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset & A_{1,\infty} = M \\ \cup_G & & \cup_L & & \cup & & & & \cup \\ A_{0,0} & \overset{H}{\subset} & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset & A_{0,\infty} = N \end{array}$$

Theorem (Bisch, C'24)

Let $N \subset P \subset M$ be an intermediate subfactor and set $B_{1,n} = A_{1,n} \cap P$. Then

$$\begin{array}{ccc} B_{1,0} & \subset & B_{1,1} \\ \cup_{G_1} & & \cup_{L_1} \\ A_{0,0} & \overset{H}{\subset} & A_{0,1} \end{array}, \qquad \begin{array}{ccc} A_{1,0} & \overset{K}{\subset} & A_{1,1} \\ \cup_{G_2} & & \cup_{L_2} \\ B_{1,0} & \subset & B_{1,1} \end{array}$$

are both non-degenerate commuting squares approximating $N \subset P$ and $P \subset M$ respectively.

⚠: G_1, G_2, L_1, L_2 are not necessarily connected.

Index $3+\sqrt{5}$

- At index $3+\sqrt{5}$ we also have the infinite depth subfactor $A_3 * A_4$, which has an intermediate subfactor with indices $\frac{3+\sqrt{5}}{2}$ and 2.
- If $G = S(3,3,3,3)$, there exist no G_i such that $G = G_1 G_2$ and

$$\|G_1\|^2 = \frac{3+\sqrt{5}}{2}, \; \|G_2\|^2 = 2.$$

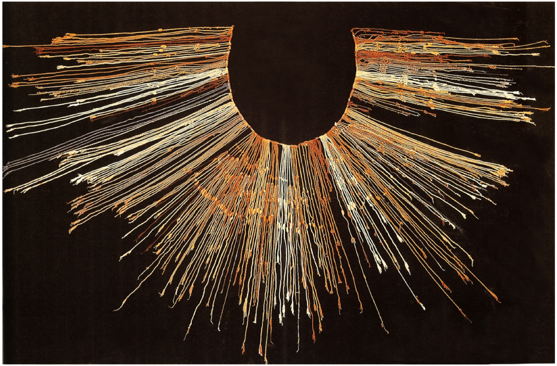
- The 1-parameter family of connections for $S(3,3,3,3)$ must produce a hyperfinite A_∞-subfactor with index $3+\sqrt{5}$.

Conclusion

There are hyperfinite A_∞-subfactors at each of these indices:

Index	constructed using...
$\frac{1}{2}(5+\sqrt{13})$	GPA embedding
≈ 4.37720	GPA embedding
$\frac{1}{2}(5+\sqrt{17})$	both
$3+\sqrt{3}$	1-parameter family
$\frac{1}{2}(5+\sqrt{21})$	1-parameter family
5	1-parameter family
$3+\sqrt{5}$	1-parameter family

Thank you very much!



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